RELATING THE MINIMAL ANNULUS WITH THE CIRCUMRADIUS OF A CONVEX SET

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Abstract. In this paper we relate the minimal annulus of a planar convex body K with its circumradius, obtaining all the upper and lower bounds, in terms of these quantities, for some of the classic geometric measures associated with the set: the diameter, the minimal width and the inradius. We prove the optimal inequalities for each one of those problems, determining also its corresponding extremal sets.

1. Introduction

Let $K \subset \mathbb{R}^2$ be a convex body (compact convex set). Associated with K there are a number of well-known functionals: the *area* $A = A(K)$ and the *perimeter* $p = p(K)$; the *diameter* $D = D(K)$ and the *minimal width* $\omega = \omega(K)$ (minimum distance between two parallel support lines of K); among all discs containing K there is exactly one (circumcircle) with minimum radius, the *circumradius* R_K of K; among all discs contained in K, those whose radii have maximum value (incircles) provide the *inradius* r_K of K.

Another interesting functional to be considered for a convex body K is the thickness of its *minimal annulus*. The minimal annulus of K is the annulus (the closed set consisting of the points lying between two concentric discs –concentric n-balls in $\mathbb{R}^n$) with minimum difference of radii that contains the boundary of K. Of course, the minimal annulus is uniquely determined (Bonnesen [2] in $\mathbb{R}^2$, Kritikos [8] in $\mathbb{R}^3$ and Bárány [1] in higher dimension). From now on, we shall denote by $A(c, r, R)$ the minimal annulus of the planar convex body K, where c, r and R represent, respectively, its center, radius of the inner circle, and radius of the outer circle. This object and its properties were studied mainly by Bonnesen for planar convex sets (see [2] and [3]). More recently, very interesting works have appeared, in which, the minimal annulus has been studied in a more general setting: for arbitrary dimension, replacing the ball by the boundary of a fixed smooth strictly convex body, in Minkowski space... (see, for instance, [1, 9, 10, 11, 12, 15]).

Another interesting problem would be to look for inequalities involving the classical functionals and the minimal annulus, finding the convex sets for which the equality sign is attained: the extremal sets. In [2], [5] and [4], Bonnesen and Favard studied this type of problems: in [2] and [5] the minimum and the maximum of the isoperimetric deficit $p^2/(4\pi) - A$, for given minimal annulus were obtained; in the third paper, the optimal bounds of the area and the perimeter for fixed minimal annulus were determined.

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In [6], the bounds for the remaining measures (diameter, minimal width, circumradius and inradius) in terms of the minimal annulus have been obtained. In [7], the problem of optimizing the classical magnitudes when the minimal annulus and the inradius are fixed is solved: let us note that if three measures are involved, the question becomes more interesting when the inequality, named optimal, provides the maximum or minimum value of a measure for *each pair* of possible values of the others.

In this paper, we obtain all the possible (and optimal) relations which state the maximum and minimum values of the diameter, the minimal width and the inradius of a convex body, when its minimal annulus and its circumradius are given. We prove the optimal inequalities for each one of these problems, determining also their corresponding extremal sets. The inequalities that state the best bounds of the area and the perimeter for fixed minimal annulus and circumradius were obtained in [6]. So, the results proved here close the problem: all the possible cases involving minimal annulus, circumradius and inradius are solved.

2. Some previous results

Before stating the main results of the paper, let us consider some properties of the minimal annulus of a convex body K, which will play a crucial role in the proofs of the results. Let us denote by c_r and C_R, respectively, the inner and the outer circles of the minimal annulus $A(c,r,R)$ of K. As usual, ∂K will denote the boundary of the set K. Given two points $P,Q\in\mathbb{R}^2$, PQ will denote the straight line determined by them; $\overline{PQ}$ the line segment joining them; and $\widehat{PQ}$ any circular arc with P,Q as extreme points. Besides, if P,Q lie on a circumference (with center c), we call *central angle* of P and Q the angle $\angle(PcQ)$ determined by them with respect to the center c.

The following well-known properties were studied by Bonnesen in [2]:

(P1) *Each one of the circumferences ∂c_r and ∂C_R touches the boundary of K in, at least, two points.*

(P2) *The sets $\partial c_r\cap\partial K$ and $\partial C_R\cap\partial K$ can not be separated.*
(Two sets A and B can be separated if there exists a line ℓ such that $A\subset\ell^+$ and $B\subset\ell^-$, where ℓ^+, ℓ^- represent the halfplanes determined by ℓ).

(P3) *The minimal annulus of a convex body K is uniquely determined.*

(P4) *The minimal annulus of a convex body K is the only annulus that contains ∂K and verifies properties* **(P1)** *and* **(P2)**.

The following lemmas were obtained in [6], where we proved some properties of the minimal annulus of a convex body K, as well as its relation with the circumradius of K. They will be very useful in the proofs of the results.

Lemma 1. *Let K be a convex body with minimal annulus $A(c,r,R)$. The following properties hold:*

(a) *There are points $P,Q\in\partial C_R\cap\partial K$ whose central angle α verifies $\alpha\geq 2\arccos(r/R)$.*

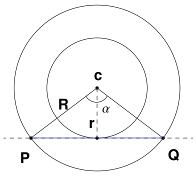

Figure 1. The limit case when the central angle of the points $P,Q\in\partial C_R\cap\partial K$ is $\alpha=2\arccos(r/R)$.

(b) *K contains a cap-body: the convex hull of c_r and two points of $\partial C_R \cap \partial K$, whose minimal annulus is $A(c, r, R)$ (a cap-body is the convex hull of a disc and countable many points such that the segment joining any pair of them intersects the disc).*

(c) *K is contained in a circular slice of C_R determined by two support lines to c_r, whose minimal annulus is $A(c, r, R)$ (a circular slice is the part of a circle bounded by two straight lines, whose intersection point, if it exists, is not interior to it).*

The following lemma collects some properties relating the minimal annulus of a convex body with its circumradius. From now on, we shall denote by C_K the circumcircle of the body K, and by x_0 its circumcenter.

Lemma 2. *Let K be a convex body with minimal annulus $A(c, r, R)$, circumcircle C_K and circumradius R_K. The following properties hold:*

(i) *$R_K \leq R$.*

(ii) *$c_r \subset K \subset C_R \cap C_K$.*

(iii) *Either $C_R \equiv C_K$, or $\partial C_K \cap \partial C_R$ has exactly two points, denoted by A and B.*

(iv) *If $C_K \not\equiv C_R$, then the points $\{A, B\} = \partial C_K \cap \partial C_R$ determine a central angle α such that $\alpha \geq 2 \arccos(r/R)$.*

(v) *The circular arc $\widehat{AB} \subset \partial C_K \subset C_R$ can not be smaller than a semi-circumference.*

(vi) *The tangent line to c_r, which is parallel and closer to the segment $\overline{AB}$, intersects ∂C_R in two points A', B', such that there exists, at least, one point $P \in \partial K \cap \partial C_R$ lying on one of the arcs $\widehat{AA'}$, $\widehat{BB'}$. Without loss of generality, let us suppose that $P \in \widehat{AA'}$. Then, there exists another point $Q \neq P$ lying on the arc $\widehat{PB}$, such that the central angle determined by P and Q verifies $\alpha \geq 2 \arccos(r/R)$, see Figure 2.*

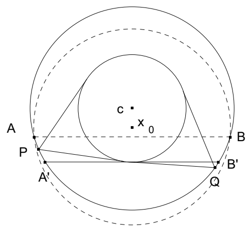

FIGURE 2. There are, at least, two points $P, Q \in \partial K \cap \partial C_R$.

(vii) *K contains the 2-cap-body $K^c = \mathrm{conv}\{c_r, P, Q\}$, with P, Q obtained from* (vi).

(viii) *The 2-cap-body K^c of the above property* (vii) *determines on the boundary of c_r two circular arcs, each one having, at least, one point of ∂K.*

(ix) *K is contained in the intersection of C_K with the circular slice of C_R determined by the support lines to c_r through the points of $\partial K \cap \partial c_r$ given by property* (viii).

From now on, we will follow the notation of the above Lemma 2: A, B will denote the intersection points of ∂C_K and ∂C_R; besides, we will denote by A' and B' the intersection points of ∂C_R with the parallel line to AB which is tangent to ∂c_r (see Figure 2).

In the following sections, we are going to obtain all the possible (and optimal) relations which state the maximum and minimum values of the diameter, the minimal width and the inradius of a convex body, when its minimal annulus and its circumradius are given.

3. Optimizing the diameter

In this section we state the relation between the minimal annulus, the circumradius and the diameter of a convex body. More precisely, we obtain the best (upper and lower) bounds for D, when the minimal annulus and the circumradius of the convex body are fixed, determining also the extremal sets in each case. We start with the upper bounds.

Theorem 1. *Let K be a convex body with minimal annulus $A(c, r, R)$ and circumradius R_K. Then, its diameter D verifies $D \leq 2R_K$. The equality holds for any set containing diametrically opposite points of ∂C_K.*

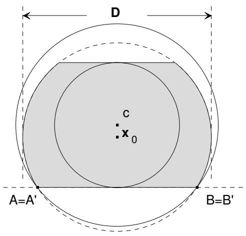

Figure 3. A convex body with maximum diameter.

Proof. The inequality $D \leq 2R_K$ always holds, independently of the minimal annulus. Now, the set shown in Figure 3 has minimal annulus $A(c, r, R)$, its circumradius is R_K and its diameter $D = 2R_K$; hence, there are sets for which the equality holds. $\square$

From now on, we will denote by N and N' the north poles of the circumferences ∂C_R and ∂C_K, i.e., the intersection points of the straight line cx_0 with ∂C_R and ∂C_K, respectively, which lie over the line segment $\overline{AB}$.

Theorem 2. *Let K be a convex body with minimal annulus $A(c, r, R)$ and circumradius R_K. Then, its diameter D verifies:*

$$D \geq R + r \qquad \text{if } \; R \leq \frac{5}{3}r \;\text{ and }\; R_K \leq \frac{R+r}{\sqrt{3}}. \tag{1}$$

The equality holds, for instance, for the cap-body $\operatorname{conv}\{c_r, A, B, N'\}$ *(see Figure 4).*

$$D \geq \sqrt{3}R_K \qquad \text{if} \quad \begin{cases} R \leq \frac{5}{3}r & \text{and} \quad R_K \geq \frac{R+r}{\sqrt{3}}, \quad \text{or} & \text{(2.a)} \\ \frac{5}{3}r \leq R \leq 2r & \text{and} \quad R_K \geq \frac{2}{\sqrt{3}}\sqrt{R^2 - r^2}. & \text{(2.b)} \end{cases} \tag{2}$$

The equality holds in both cases, for instance, for the cap-body $\operatorname{conv}\{c_r, A, B, N'\}$*, when* $\triangle(ABN')$ *is an equilateral triangle (see Figure 5).*

$$D \geq 2\sqrt{R^2 - r^2} \qquad \text{if} \quad \begin{cases} \frac{5}{3}r \leq R \leq 2r \quad \text{and} \quad R_K \leq \frac{2}{\sqrt{3}}\sqrt{R^2 - r^2}, \quad \text{or} & \text{(3.a)} \\ 2r \leq R. & \text{(3.b)} \end{cases} \tag{3}$$

In (3.a), equality holds, for instance, for the cap-body $\operatorname{conv}\{c_r, A, B, N'\}$*; in (3.b), for the convex body* $\operatorname{conv}\{c_r, A, B, Z\}$*, where $Z \neq A$ is the intersection point of ∂C_K and the circumference with center B and radius $d(A, B) = 2\sqrt{R^2 - r^2}$ (see Figure 6).*

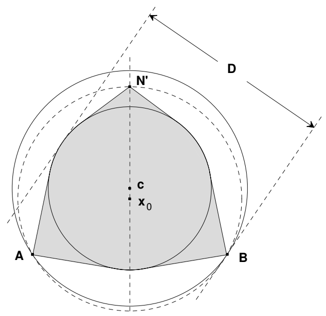

FIGURE 4. Set with minimum diameter for $R \le 5r/3$, $R_K \le (R+r)/\sqrt{3}$.

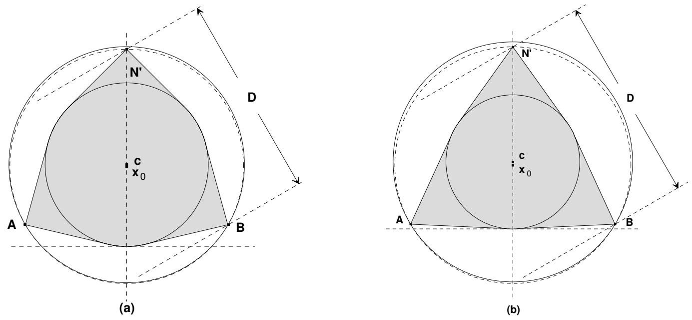

FIGURE 5. Sets with minimum diameter when (a) $R \le 5r/3$, $R_K \ge (R+r)/\sqrt{3}$, and (b) $5r/3 \le R \le 2r$, $R_K \ge 2\sqrt{R^2-r^2}/\sqrt{3}$.

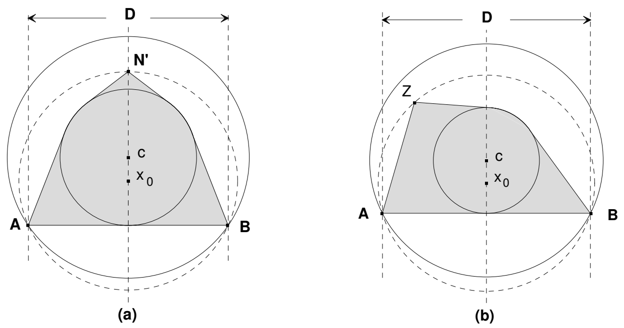

FIGURE 6. Sets with minimum diameter when (a) $5r/3 \le R \le 2r$, $R_K \le 2\sqrt{R^2-r^2}/\sqrt{3}$, and (b) $2r \le R$.

Let us note that the extremal set $\mathrm{conv}\{c_r, A, B, Z\}$ for inequality (3.b) is not always a cap-body, since the line segment $\overline{AZ}$ can have no intersection with c_r (see Figure 6(b)).

Proof. We develop the proof in different steps: first, we see that all the inequalities hold; then, we will show that they are optimal, determining the extremal sets.

(I) THE INEQUALITIES. Let us suppose first that $R \le 5r/3$ and $R_K \le (R+r)/\sqrt{3}$. In [6, Proposition 3], the relation between the minimal annulus and the circumradius was stated. It was proved that when $R \le 5r/3$, it always holds $D \ge R + r$, for any (possible) value of R_K. Besides, it is well-known that if K is a convex body with circumradius R_K, then $D \ge \sqrt{3}R_K$ (see, for instance, [3, p. 84]). Hence, we can assure that

$$D \ge \max\bigl\{R + r, \sqrt{3}R_K\bigr\} = R + r,$$

since, by hypothesis, $\sqrt{3}R_K \le R + r$. It gives the lower bound in inequality (1). Now, if $R \le 5r/3$ but $R_K \ge (R + r)/\sqrt{3}$, then $D \ge \max\bigl\{R + r, \sqrt{3}R_K\bigr\} = \sqrt{3}R_K$, which states the bound in (2.a).

Let us suppose now that $R \in \bigl[5r/3, 2r\bigr]$ and $R_K \ge 2\sqrt{R^2 - r^2}/\sqrt{3}$. Since $R \ge 5r/3$, it is known (see [6, Proposition 3]) that $D \ge 2\sqrt{R^2 - r^2}$. Hence,

$$D \ge \max\left\{2\sqrt{R^2 - r^2}, \sqrt{3}R_K\right\} = \sqrt{3}R_K,$$

which proves the lower bound for (2.b). If, on the contrary, $R_K \le 2\sqrt{R^2 - r^2}/\sqrt{3}$, then $D \ge \max\bigl\{2\sqrt{R^2 - r^2}, \sqrt{3}R_K\bigr\} = 2\sqrt{R^2 - r^2}$, inequality (3.a).

Finally, let us suppose that $R \ge 2r$. Then, in particular, $R \ge 5r/3$, which assures that $D \ge 2\sqrt{R^2 - r^2}$ (see again [6, Proposition 3]). Hence, $D \ge \max\bigl\{2\sqrt{R^2 - r^2}, \sqrt{3}R_K\bigr\}$. If $\sqrt{3}R_K \ge 2\sqrt{R^2 - r^2}$, using the trivial inequality $R \ge R_K$, we would get $3R^2 \ge 4(R^2 - r^2)$, or equivalently, $R \le 2r$, a contradiction. Therefore, the above maximum is $2\sqrt{R^2 - r^2}$, which shows inequality (3.b).

In order to conclude the proof of the theorem, we have to show that these bounds are best possible; i.e., we have to determine the families of extremal sets for each of them. First, we distinguish the particular case $R = R_K$.

(ii) The particular case $R_K = R$. It is an easy computation to check that inequalities (1), (2) and (3) are reduced to

$$D \ge R + r \quad \text{if} \quad R \le \frac{1 + \sqrt{3}}{2}r, \tag{4}$$

$$D \ge \sqrt{3}R \quad \text{if} \quad \frac{1 + \sqrt{3}}{2}r \le R \le 2r, \tag{5}$$

$$D \ge 2\sqrt{R^2 - r^2} \quad \text{if} \quad 2r \le R. \tag{6}$$

There are many families of sets for which the equality holds in inequalities (4) and (5): the well-known constant width sets verify $D = R + r$ when $R \le (\sqrt{3} + 1)r/2$, since their circumcircle and incircle are always concentric, and hence determine their minimal annulus; the so called Yamanouti sets verify $D = \sqrt{3}R$ when $(\sqrt{3} + 1)r/2 \le R \le 2r$, again because the circumcircle and the incircle are concentric, and determine the minimal annulus (a Yamanouti set is the convex hull of an equilateral triangle and three circular arcs with center on each vertex of the triangle and radius not greater than its side length).

Now, let us suppose that $R \ge 2r$. Let $A \equiv A'$ and $B \equiv B'$, and we consider the circular sector ABM, where M is the intersection point of ∂C_R and the circumference with center B and radius $d(A, B) = 2\sqrt{R^2 - r^2}$ (see Figure 7).

Clearly, the straight lines AB and BM support c_r, and the contact points can not be separated from $\{A, B, M\}$; hence, the set has minimal annulus $A(c, r, R)$. Its circumradius is R, since A, B, M determine an acute-angled triangle. Finally, since $R \ge 2r$, the point M lies on the circular arc $\widehat{AN} \subset \partial C_R$. Therefore, $d(A, M) \le d(B, M) = d(A, B)$, which assures that the diameter is $D = d(B, M) = d(A, B) = 2\sqrt{R^2 - r^2}$.

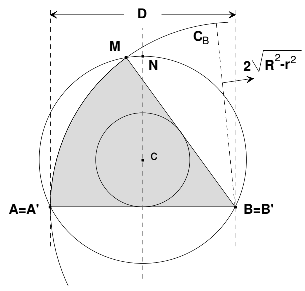

FIGURE 7. The extremal set for $R_K = R$ and $R \geq 2r$.

From now on, we will assume that $R_K < R$, i.e., that $C_K \not\equiv C_R$.

(III.A) THE EXTREMAL SETS FOR INEQUALITY (1). Let R, r be given such that $R \leq 5r/3$. In this case, the distance $d(A', B') = 2\sqrt{R^2 - r^2} \leq R+r$. Let us take $A \equiv A'$ and $B \equiv B'$, and let us consider the circles C_A and C_B, both with radius $R + r$, and centers A and B, respectively. Then, ∂c_r touches the circumferences ∂C_A and ∂C_B in the intersection points M_A, M_B of the straight lines Ac and Bc with ∂c_r, respectively (see Figure 8(a)).

If R_K is such that $d(A, N') = d(B, N') \leq R + r$ (i.e., if N' lies inside the circle C_A –and C_B), then $L = \text{conv}\{c_r, A, B, N'\}$ is contained in the intersection of $C_A \cap C_B$ with the closed half-plane determined by AB (see Figure 8). Since $A \equiv A'$ and $B \equiv B'$, then x_0 lies over the segment $\overline{AB}$, which assures that $\triangle(AN'B)$ is an acute-angled triangle; hence, L has circumradius R_K. By property **(P4)**, its minimal annulus is $A(c, r, R)$.

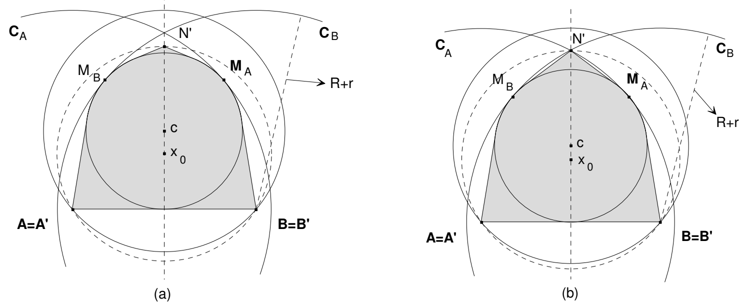

FIGURE 8. L has minimum diameter when $R_K \leq \sqrt{(R+r)^3/(8r)}$.

Let us study the diameter of these figures. Since $R \leq 5r/3 < 2r$, the line segments $\overline{N'A}$ and $\overline{N'B}$ always intersect c_r (the limit case corresponds to $N' \equiv N$ and $R = 2r$); therefore, $M_A, M_B \in \partial L$. It follows that $D(L) = d(A, M_A) = d(B, M_B) = R + r$, since we have assumed that $d(A, N') = d(B, N') \leq R + r$ and $R \leq 5r/3$ (which implies $d(A, B) = 2\sqrt{R^2 - r^2} \leq R + r$). An easy computation shows that $N' \in \partial C_A \cap \partial C_B$ if and only if $R_K = \sqrt{(R+r)^3/(8r)}$. Thus, the above construction for the set L can be developed only if $R_K \leq \sqrt{(R+r)^3/(8r)}$ (see Figure 8).

However, from such a value of R_K, $d(A, N') = d(B, N') > R + r$, and the above construction does not work. So, let us suppose that $R \leq 5r/3$ and $R_K > \sqrt{(R+r)^3/(8r)}$, which implies $d(A',B') = 2\sqrt{R^2-r^2} \leq R+r < d(A',N') = d(B',N')$. Let us choose the circumcenter x_0 such that $A \equiv A'$ and $B \equiv B'$. Then, moving x_0 on the line cx_0 far away from c, we increase the distance $d(A,B)$ (now $A \not\equiv A'$, $B \not\equiv B'$), decreasing $d(A,N') = d(B,N')$ at the same time. By continuity, there is a position of x_0 for which $d(A,B) = d(A,N') = d(B,N')$; i.e., such that $A,B,N' \in \partial C_K$ form an equilateral triangle. Then, $d(A,B) = d(A,N') = d(B,N') = \sqrt{3}R_K$, and the set $L = \operatorname{conv}\{c_r, A, B, N'\}$ has circumradius R_K and minimal annulus $A(c,r,R)$ (see Figure 9).

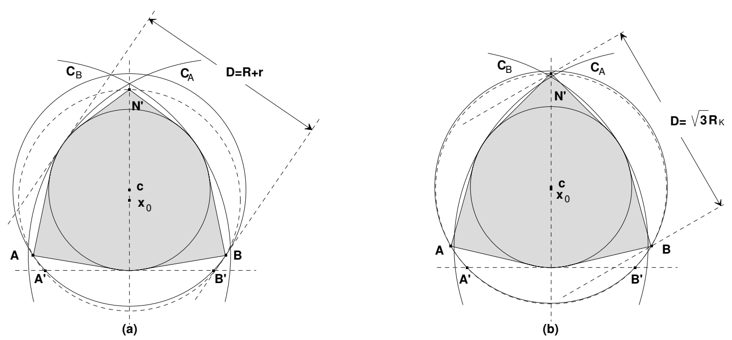

FIGURE 9. (a) $D(L) = R+r$, if $R \leq 5r/3$ and $\sqrt{(R+r)^3/(8r)} \leq R_K \leq (R+r)/\sqrt{3}$; (b) $D(L) = \sqrt{3}R_K$, if $R \leq 5r/3$ and $R_K \geq (R+r)/\sqrt{3}$.

Clearly, $D(L) \geq d(A,B)$. Thus, two different cases appear:

- If $N' \in C_A \cap C_B$ (i.e., if $L \subset C_A \cap C_B$), then $R+r \geq d(A,B) = \sqrt{3}R_K$ (see Figure 9(a)); so, $D(L) = R+r$.
- If $N' \notin C_A \cap C_B$, then $\sqrt{3}R_K = d(A,B) \geq R+r$, and the diameter is $\sqrt{3}R_K$ (see Figure 9(b)).

In short, if $\sqrt{(R+r)^3/(8r)} < R_K \leq (R+r)/\sqrt{3}$, the set $L = \operatorname{conv}\{c_r, A, B, N'\}$ shown in Figure 9(a) is extremal for inequality (1); it concludes the proof of this inequality.

(III.B) THE EXTREMAL SETS FOR INEQUALITY (2). The previous argument also shows that if $R_K \geq (R+r)/\sqrt{3}$ (and $R \leq 5r/3$), then the analogous set L, shown in Figure 9(b), is extremal for inequality (2.a).

So, let us suppose that $5r/3 \leq R \leq 2r$ and $R_K \geq 2\sqrt{R^2-r^2}/\sqrt{3}$. The points A', N', B' determine an isosceles triangle, with side lengths

$$d(A',B') = 2\sqrt{R^2-r^2}, \quad d(A',N') = d(B',N') = \left[2R_K\left(R_K + \sqrt{R_K^2 - R^2 + r^2}\right)\right]^{1/2}.$$

An easy computation shows that $\triangle(A'N'B')$ is an equilateral triangle if and only if $\sqrt{3}R_K = 2\sqrt{R^2-r^2}$, and also that $d(A',B') \leq d(A',N') = d(B',N')$ if and only if $\sqrt{3}R_K \geq 2\sqrt{R^2-r^2}$, our hypothesis. Hence, $d(A',B') \leq d(A',N') = d(B',N')$.

Let us choose again the circumcenter x_0 such that $A \equiv A'$ and $B \equiv B'$. Then, moving x_0 on the line cx_0 far away from c, we increase the distance $d(A,B)$, decreasing $d(A,N') = d(B,N')$ at the same time. By continuity, there exists a position of x_0 for which $d(A,B) = d(A,N') = d(B,N')$; i.e., such that $\triangle(AN'B)$ is an equilateral triangle. In this case, $d(A,B) = d(A,N') = d(B,N') = \sqrt{3}R_K$, and the convex body

$L = \operatorname{conv}\{c_r, A, B, N'\}$ has circumradius R_K. Since $R \le 2r$, the sides of the triangle $\triangle(AN'B)$ intersect c_r, which implies that the contact points of ∂L with ∂C_R and ∂c_r, respectively, can not be separated: L has minimal annulus $A(c, r, R)$ (see Figure 10).

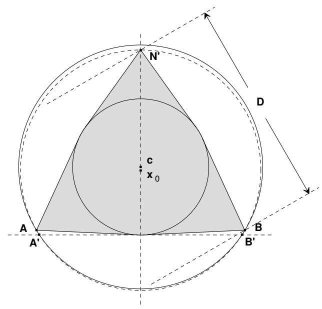

FIGURE 10. $D(L) = \sqrt{3}R_K$, if $5r/3 \le R \le 2r$ and $R_K \ge 2\sqrt{R^2 - r^2}/\sqrt{3}$.

The diameter of L is, either the diameter of $\triangle(AN'B)$, i.e., $\sqrt{3}R_K$, or the distance from any vertex to a support line of the opposite circular arc of ∂c_r, i.e., $R+r$. Since $5r/3 \le R$, then $2\sqrt{R^2 - r^2} \ge R + r$, and from $R_K \ge 2\sqrt{R^2 - r^2}/\sqrt{3}$, we obtain $\sqrt{3}R_K \ge R + r$; hence $D(L) = \sqrt{3}R_K$ (see Figure 10). It concludes the proof of inequality (2.b).

(III.C) THE EXTREMAL SETS FOR INEQUALITY (3). Let us suppose that $5r/3 \le R \le 2r$ and $R_K \le 2\sqrt{R^2 - r^2}/\sqrt{3}$. We take $A \equiv A'$ and $B \equiv B'$, and let C_A, C_B be the circles with radius $2\sqrt{R^2 - r^2} = d(A, B)$ and centers A and B, respectively.

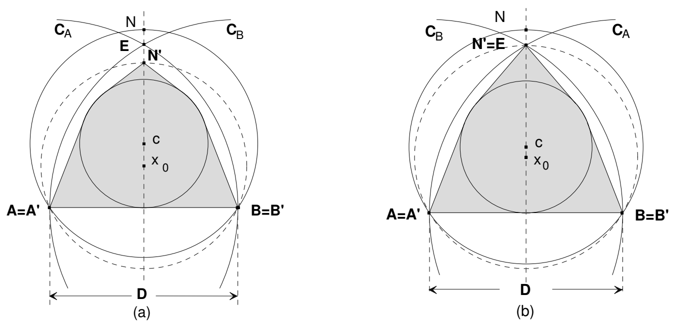

FIGURE 11. (a) $\operatorname{conv}\{c_r, A, B, N'\}$ has minimum diameter if $5r/3 \le R \le 2r$ and $R_K \le 2\sqrt{R^2 - r^2}/\sqrt{3}$. (b) The limit case $R_K = 2\sqrt{R^2 - r^2}/\sqrt{3}$.

An easy computation shows that $d(A, N) = \sqrt{2R(R + r)}$; using that $R \le 2r$, we have $d(A, N) = d(B, N) \ge d(A, B) = 2\sqrt{R^2 - r^2}$. Hence, $\partial C_A \cap \partial C_B$ gives a point $E \in C_R$. Let us note that for a value of R_K such that the point N' verifies $d(N', c) \le d(E, c)$, the set $L = \operatorname{conv}\{c_r, A, B, N'\}$ is the required solution (see Figure 11): its circumradius is R_K because A, N', B do not lie on the same semi-circumference; the minimal annulus is $A(c, r, R)$, because $\partial\triangle(AN'B) \cap c_r \ne \emptyset$; finally, since $L \subset C_A \cap C_B$, $D(L) = 2\sqrt{R^2 - r^2}$.

It is easy to see that $d(N', c) = d(E, c)$, i.e., $N' \equiv E$, only if $R_K = 2\sqrt{R^2 - r^2}/\sqrt{3}$; so, $d(N', c) \le d(E, c)$ when $R_K \le 2\sqrt{R^2 - r^2}/\sqrt{3}$, our hypothesis. It shows inequality (3.a).

Finally, let us suppose that $R \geq 2r$, for any (possible) value of R_K. Let us take $A \equiv A'$ and $B \equiv B'$. From $R \geq 2r$, it follows that $d(A, N) = d(B, N) \leq d(A, B) = 2\sqrt{R^2 - r^2}$. Hence, ∂C_A and ∂C_B intersect in the exterior of C_R (see Figure 12), and $\partial C_B \cap \partial C_R$ gives a point $M \in \widehat{AN}$, such that the line segment $\overline{MB}$ supports c_r. On the other hand, there is a point $Z \in \partial C_B \cap \partial C_K$ lying on $\widehat{AM} \subset \partial C_B$ (if $\overline{AB}$ is a diameter-chord of C_K, then $Z \equiv A$, as shown in Figure 12(b)).

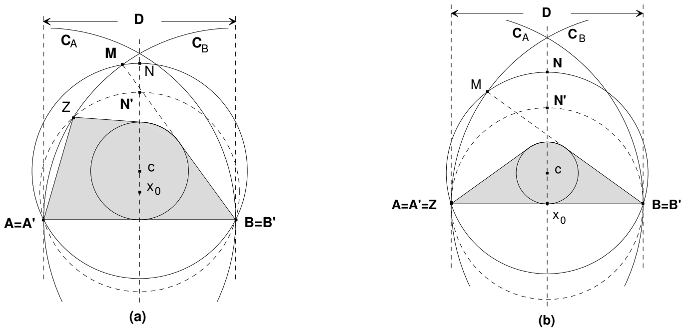

FIGURE 12. $L = \text{conv}\{c_r, A, B, Z\}$ has minimum diameter if $R \geq 2r$.

The convex body $L = \text{conv}\{c_r, A, B, Z\}$ provides the required solution: it has circumradius R_K (A, B, Z do not lie on the same semi-circumference) and minimal annulus $A(c, r, R)$. Finally, since L is contained in the circular sector ABM, and contains the center B and two points A, Z of the circular arc, its diameter is $D(L) = 2\sqrt{R^2 - r^2}$ (see Figure 12). It concludes the proof of inequality (3.b), and the theorem.

Let us note that, in the last case, the set L can not be usual $\text{conv}\{c_r, A, B, N'\}$, because for certain values of R, r, R_K, the line segments $\overline{AN'}$ and $\overline{BN'}$ do not touch ∂c_r (see Figure 12(b)); then, $A(c, r, R)$ can not be the minimal annulus of the set. □

4. Optimizing the minimal width and the inradius

In this section we state the relation between the minimal annulus, the circumradius and both, the minimal width and the inradius of a convex body K. More precisely, we are going to obtain the best bounds (upper and lower bounds) for ω and r_K, when the minimal annulus and the circumradius of the convex body are fixed, determining also the extremal sets in each case. The results for both cases, the minimal width and the inradius, can be proved in a similar way. So, we will state them together.

Theorem 3. *Let K be a convex body with minimal annulus $A(c, r, R)$ and circumradius R_K. Then, its minimal width ω and its inradius r_K verify*

$$\omega \geq 2r \quad \text{and} \quad r_K \geq r.$$

The equality holds for any set containing diametrically opposite points of ∂c_r.

Proof. In [6] it was proved that the above inequalities always hold, independently of the value of R_K. Therefore, it suffices to show that, for any possible value of R_K, there exists a convex body with minimal annulus $A(c, r, R)$ and, for each case, minimal width $\omega = 2r$ or inradius $r_K = r$. For instance, the set in Figure 13 verifies the required conditions. □

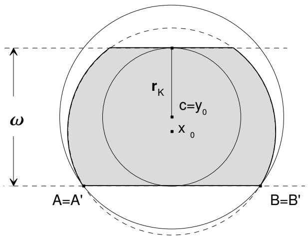

FIGURE 13. A set with minimal width $\omega = 2r$ and inradius $r_{\kappa} = r$.

Before stating the opposite bound, let us construct the following set: for $A(c, r, R)$ and R_K given, let us consider the circle C_K with radius R_K such that the straight line AB supports c_r. Let ℓ denote the tangent line to ∂c_r, passing through the point A (see Figure 14). We define the *asymmetric circular wedge*, denoted by $K^{\angle}$, as the intersection of C_K with the circular slice of C_R determined by the straight lines AB and ℓ.

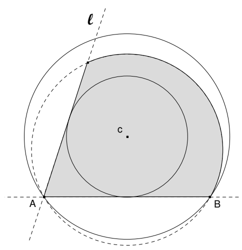

FIGURE 14. Asymmetric circular wedge $K^{\angle}$.

Theorem 4. *Let K be a convex body with minimal annulus $A(c, r, R)$ and circumradius R_K. Then, its minimal width ω verifies:*

$$\omega \leq R_K + \sqrt{R_K^2 - R^2 + r^2}$$

(7)
$$\text{if} \begin{cases} R \leq 2r, \quad \text{or} & \text{(7.a)} \\ 2r \leq R \leq r\sqrt{2(2+\sqrt{2})} \;\; \text{and} \;\; R_K \leq \dfrac{R^4}{4r(R^2 - 2r^2)}. & \text{(7.b)} \end{cases}$$

$$\omega \leq \frac{4r(R^2 - r^2)}{R^4}\left(R^2 - 2r^2 + 2r\sqrt{R_K^2 - R^2 + r^2}\right)$$

(8)
$$\text{if} \begin{cases} 2r \leq R \leq r\sqrt{2(2+\sqrt{2})} \;\; \text{and} \;\; R_K \geq \dfrac{R^4}{4r(R^2 - 2r^2)}, \quad \text{or} & \text{(8.a)} \\ R \geq r\sqrt{2(2+\sqrt{2})}. & \text{(8.b)} \end{cases}$$

And its inradius r_{κ} verifies

$$r_{\kappa} \leq 2r\left(1 - r\frac{R_K - \sqrt{R_K^2 - R^2 + r^2}}{R^2 - r^2}\right). \tag{9}$$

The equality holds, in all cases, for the asymmetric circular wedge $K^{\angle}$ (see Figure 15).

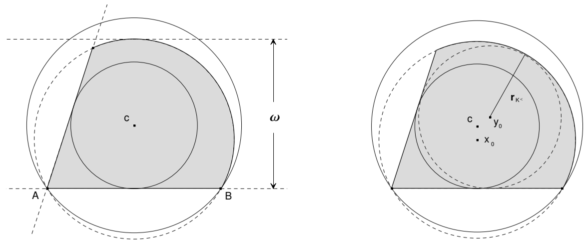

FIGURE 15. $K^{\angle}$ has maximum minimal width and inradius.

Proof. Let us note that, if $R_K = R$, then inequalities (7), (8) and (9) can be written as

$$\omega \leq \begin{cases} R + r & \text{if } R \leq 2r, \\ \omega \leq \dfrac{4r}{R^2}(R^2 - r^2) & \text{if } R \geq 2r \end{cases} \qquad \text{and} \qquad r_{\scriptscriptstyle K} \leq \frac{2rR}{R+r},$$

respectively. In [6, Propositions 2 and 7], it was proved that these relations hold for the minimal width and the inradius when the minimal annulus is prescribed. Thus, from now on we can suppose that $R_K < R$, and hence, that $C_K \not\equiv C_R$.

Property (ix) of Lemma 2 assures that K is contained in the intersection of C_K with the circular slice of C_R determined by the support lines to c_r in two suitable points of $\partial c_r \cap \partial K$, which are separated by the line segment $\overline{AB}$; we denote by K_1 this kind of sets. Besides, by property (vi) of this lemma, we know that at least one of the above support lines intersects ∂C_R in a point P, lying either on the circular arc $\widehat{AA'}$, or on $\widehat{BB'}$; we can suppose, for instance, that $P \in \widehat{AA'}$ (see Figure 16). Therefore, $\omega \leq \omega(K_1)$ and $r_{\scriptscriptstyle K} \leq r_{\scriptscriptstyle K_1}$, and the problem is reduced to consider this particular family of sets.

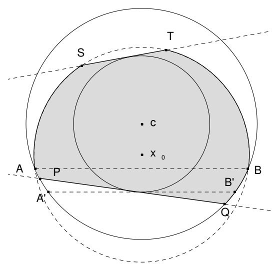

FIGURE 16. Reducing the problem to the sets K_1.

Following the notation of Figure 16, we represent by $S, T \in \partial C_K$ and $Q \in \partial C_R$ the intersection points (besides P), of ∂C_K and ∂C_R with the straight lines determining K_1.

For each fixed segment $\overline{PQ}$, both the minimal width and the inradius of K_1 are minimum (respectively, $2r$ and r) when ST is parallel to $\overline{PQ}$. If we move ST continuously on ∂c_r in the anti-counter-clockwise, we obtain all the possible sets K_1. Let us note that the width in the orthogonal direction to PQ is given, depending on the relation between r, R and R_K, by the distance, to $\overline{PQ}$, either from the point T, or from the tangent line to ∂C_K, which is parallel to $\overline{PQ}$. And this one is the direction in which the minimal width of K_1 is attained. Of course, the greater the angle determined by PQ and ST, the greater the minimal width and the inradius of K_1; therefore, the set K_1 with maximum width and inradius is obtained when the points P and S coincide (see Figure 17).

If we move ST in the counter-clockwise, we can conclude analogously that the set has maximum minimal width and inradius when $T \equiv Q$. However, this figure has both, less minimal width and less inradius than the previous one (when $P \equiv S$). In fact, let us note that the point P lies over the line segment $A'B'$, and consequently, Q lies below it; then, $d(Q, x_0) \leq d(P, x_0)$. Besides, the angles $\measuredangle(PQS) = \measuredangle(TPQ)$ when $T \equiv Q$ or $P \equiv S$, because $P, Q \in \partial C_R$ and the lines determining these angles support c_r. Therefore, the length of the arc $\overset{\frown}{AS}$, when $T \equiv Q$, is less than the length of $\overset{\frown}{TB}$, if $P \equiv S$; it implies that both the minimal width and the inradius are maximized when $P \equiv S$.

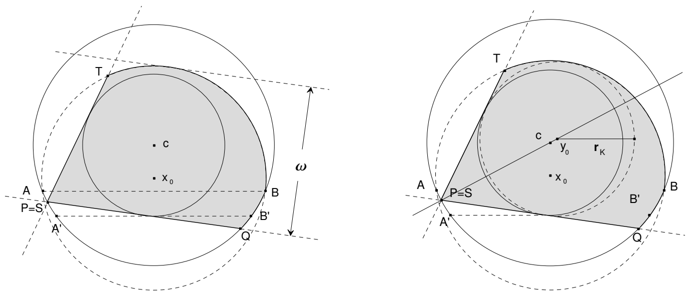

FIGURE 17. Reducing the problem to the sets K_2.

Let K_2 be this last set (see Figure 17). Then, $\omega \leq \omega(K_1) \leq \omega(K_2)$ and $r_K \leq r_{K_1} \leq r_{K_2}$. Since $P \in \partial C_R$ and the lines PT, PQ support c_r, then the angle $\measuredangle(TPQ)$ is always the same for any point P. Besides, the greater the length of the arc $\overset{\frown}{TB}$, the greater

(1) the distance between PQ and its parallel line, tangent to $\overset{\frown}{TB}$, and so the minimal width,
(2) the radius of the incircle.

For fixed P, continuously moving the circumcenter x_0 on the straight line x_0c towards c, then the part of C_K contained in C_R is bigger; hence, the length of the arc $\overset{\frown}{TB}$ increases, and hence the minimal width and the inradius. We can do this movement till $P \equiv A$. Thus, it suffices to consider the sets K_2 such that the lines determining them intersect on A (see Figure 18, left).

Finally, it is easy to see that, since the angle in A is always the same wherever A is placed, both the minimal width and the inradius will be maximal when $A \equiv A'$ (see Figure 18), this is, when the set is an asymmetric circular wedge $K^{\angle}$.

A tedious calculation shows that

$$r_{K^{\angle}} = 2r\left(1 - r\frac{R_K - \sqrt{R_K^2 - R^2 + r^2}}{R^2 - r^2}\right),$$

which states inequality (9).

We just have to compute the minimal width of $K^{\angle}$, which depends on the relation between R, r and R_K. Again, N' will denote the intersection point of the straight line cx_0 and ∂C_K, as shown in Figure 18.

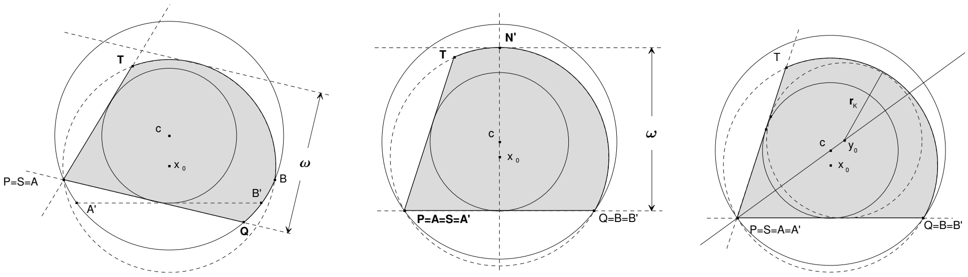

FIGURE 18. $K^{\angle}$ has maximal inradius and minimal width.

If $R\geq r\sqrt{2(2+\sqrt{2})}$, then it is easy to see that, for any possible value of R_K, the point T lies on the circular arc $\overset{\frown}{N'B}\subset\partial C_K$ (see Figure 19, left). Hence, the minimal width is the distance from T to the line segment $\overline{AB}$:

$$\omega(K_2)=\frac{4r(R^2-r^2)}{R^4}\left(R^2-2r^2+2r\sqrt{R_K^2-R^2+r^2}\right).$$

Besides, if $R=r\sqrt{2(2+\sqrt{2})}$ and $R_K=\sqrt{R^2-r^2}$, the circumcenter x_0 lies on the line segment $\overline{AB}$, and then, $T\equiv N'$ (see Figure 19, middle).

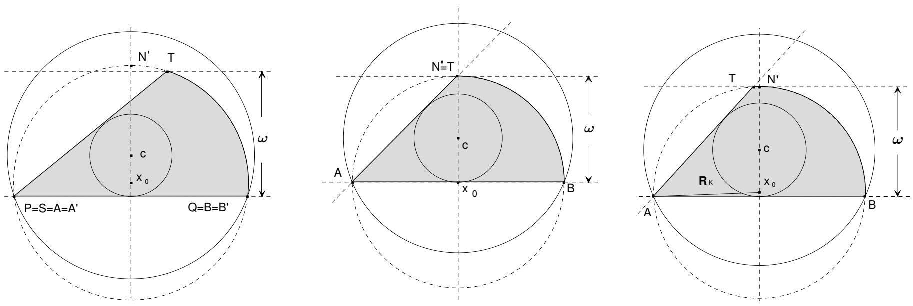

FIGURE 19. Different positions for the point $T\in\partial C_K$.

In the case $2r\leq R\leq r\sqrt{2(2+\sqrt{2})}$, $T\equiv N'$ only if $R_K=R^4/(4r(R^2-2r^2))$. Hence, if $R_K\geq R^4/(4r(R^2-2r^2))$, then T lies again on the circular arc $\overset{\frown}{N'B}$; on the contrary, if $R_K\leq R^4/(4r(R^2-2r^2))$, then $T\in\overset{\frown}{AN'}$, and the minimal width is the distance from N' to $\overline{AB}$ (see Figure 19, right):

$$\omega(K_2)=R_K+\sqrt{R_K^2-R^2+r^2}.$$

Finally, if $R\leq 2r$, the point T always lies on the arc $\overset{\frown}{AN'}$, for any possible value of the circumradius, and hence, the maximum minimal width is the distance between N' and the segment $\overline{AB}$ (see Figure 18, middle). □

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